

Homotopy Classification of Multiaxial Actions

Sylvain Cappell*

Courant Institute, New York University

Shmuel Weinberger[†]

University of Chicago

Min Yan[‡]

Hong Kong University of Science and Technology

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1 Multiaxial Action

The concept of multiaxial manifold was introduced and studied in [4, 5, 6]. For the purpose of this paper, we modify the definition as follows.

Definition. A *multiaxial* $U(n)$ -manifold is a manifold M with locally linear $U(n)$ -action, such that any isotropy group is conjugate to a unitary subgroup $U(i)$, and the action has free points.

The condition is weaker than required by the earlier work, except the existence of free points. In fact, our work also covers the case of no free points. See the discussion after the proof of Proposition 1.2. The condition is added only for the simplicity of presentation.

Our multiaxial $U(n)$ -manifold is stratified

$$M = M_0 \supset M_{-1} \supset M_{-2} \supset \cdots \supset M_{-n}, \quad (1)$$

where $M_{-i} = U(n)M^{U(i)}$ consists of all the points fixed by some subgroup conjugate to $U(i)$. The quotient $\bar{M} = M/U(n)$ is then stratified with strata

$$\bar{M}_{-i} = M_{-i}/U(n).$$

The pure strata of M are $M^{-i} = M_{-i} - M_{-i-1}$, which consists of points with isotropy group conjugate to $U(i)$. The pure strata of \bar{M} are $\bar{M}^{-i} = \bar{M}_{-i} - \bar{M}_{-i-1}$.

The isotropy groups of multiaxial manifolds have special properties.

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Proposition 1.1. *Suppose $H \subset K \subset G = U(n)$ are unitary subgroups. Then the NH -action on $(G/K)^H$ is transitive. In other words, if $H \subset K$ and $g^{-1}Hg \subset K$, then $g = \nu k$ for some $\nu \in NH$ and $k \in K$.*

Proof. The unitary group $G = U(n)$ consists of the unitary transformations of \mathbb{C}^n . The unitary subgroups K and H consist of those unitary transformations that respectively fix some subspaces V_K and V_H . Now $H \subset K$ means $V_K \subset V_H$ and $g^{-1}Hg \subset K$ means $gV_K \subset V_H$. Therefore there is a unitary transformation ν that preserves V_H and restricts to g on V_K . Then $\nu^{-1}g$ preserves V_K , so that $\nu^{-1}g \in K$. Moreover, the fact that ν preserves V_H means that $\nu \in NH$.

The transitivity of the NH -action on $(G/K)^H$ means that if $gK \in (G/K)^H$, then $gK = \nu K$ for some $\nu \in NH$. Since $gK \in (G/K)^H$ means $g^{-1}Hg \subset K$, and $gK = \nu K$ means $g = \nu k$ for some $k \in K$, we see that the transitivity is the same as the group theoretical property above. \square

Proposition 1.2. *Suppose G acts on X , such that every pair of isotropy groups satisfy the property in Proposition 1.1. Then $GX^H/G = X^H/NH$ for any isotropy group H .*

Proof. We always have the natural surjective map $X^H/NH \rightarrow GX^H/G$. Over a point in GX^H/G represented by $x \in X^H$, the fibre of the map is $(Gx)^H/NH$. Therefore the map is one-to-one if and only if the action of NH on $(Gx)^H = (G/G_x)^H$ is transitive. \square

Propositions 1.1 and 1.2 imply that

$$\bar{M}_{-i} = M_{-i}/U(n) = M^{U(i)}/U(n-i), \quad (2)$$

where $U(n-i)$ is naturally identified with the Weyl group $NU(i)/U(i)$ of $U(i)$ in $U(n)$. The equality shows that, as far as the orbit space is concerned, the study of the stratum \bar{M}_{-i} is equivalent to the study of the orbit space $M^{U(i)}/U(n-i)$ of $M^{U(i)}$, which is itself a multiaxial $U(n-i)$ -manifold. For example, if M does not have free points, then the minimal isotropy group is conjugate to $U(m)$ for some $m > 0$. Then the study of the $U(n)$ -manifold is the same as the study of the $U(n-m)$ -manifold $M^{U(m)}$, for which there are free points. This is the reason the existence of free points is added to our definition.

The equality (2) can be considered as some kind of “hereditary property” for the orbit space. We wish to have the same property for the links between various strata. In general, if G acts locally linearly on a manifold M , then the strata of M are GM^K for various isotropy groups K , and the strata of the orbit space M/G are GM^K/G . At a point $x \in M$ with isotropy group K , the normal direction of GM^K in M is a K -representation ν_x . The link of the stratum GM^K/G in M/G is then $S(\nu_x)/K$, where $S(\nu_x)$ is the unit ball of ν_x .

The link $S(\nu_x)/K$ is also stratified, with strata $KS(\nu_x)^H/K$ for various isotropy groups $H \subset K$. On the other hand, we have the inclusion of strata

$$GM^H \subset GM^K \subset M,$$

and the corresponding strata for the orbit space. The “hereditary property” we wish to have is that $KS(\nu_x)^H/K$ is the link of GM^H/G in GM^K/G .

Proposition 1.3. *Suppose $H \subset K \subset G$ satisfy the property in Proposition 1.1. Suppose S is a K -space and $X = G \times_K S$. Then*

$$1. X^H = NH \times_{NH \cap K} S^H.$$

$$2. GX^H = GS^H.$$

Proof. It is easy to see that $NH \times_{NH \cap K} S^H$ is embedded in X and fixed by H . Conversely, an element $x = gs \in X$, $g \in G$, $s \in S$, is fixed by H if and only if s is fixed by $g^{-1}Hg$. However, we have $S \cap aS \neq \emptyset$ if and only if $a \in K$. Therefore $g^{-1}Hg \subset K$. By the property in Proposition 1.1, we have $g = \nu k$ for some $\nu \in NH$ and $k \in K$. Therefore $\nu^{-1}x = ks \in S$, so that $x = \nu(\nu^{-1}x) \in NH \times_{NH \cap K} S^H$.

It is clear that $S^H \subset X^H$. The first part also implies $X^H \subset GS^H$. The second part then follows. \square

Proposition 1.4. *Suppose $H \subset K$ are isotropy groups of locally linear G -manifold M , such that the property in Proposition 1.1 is satisfied. If ν_x is the normal representation of GM^H at a point x of isotropy group H , then the link of GM^H/G in GM^K/G is*

$$KS(\nu_x)^H/K = K \times_{NH \cap K} S(\nu_x)^H/K = S(\nu_x)^H/(NH \cap K).$$

Proof. The classical slice theorem says that, there is a slice σ_x at x , which is a K -representation, such that $G \times_K \sigma_x$ is a tube neighborhood of the orbit Gx . Since the study of link between strata is local, we may assume $M = G \times_K \sigma_x$ is the tube. Then $GM^K = G \times_K \sigma_x^K = G/K \times \sigma_x^K$, and the link of $GM^K/G = \sigma_x^K$ in $M/G = \sigma_x/K$ is $S(\nu_x)/K$ for the K -representation $\nu_x = \sigma_x/\sigma_x^K$.

Given that the isotropy groups $K \subset H$ satisfy the property in Proposition 1.1, we may apply Proposition 1.3 to get the corresponding strata

$$GM^K = G/K \times \sigma_x^K \subset GM^H = G \times_{NH \cap K} \sigma_x^H \subset M = G \times_K \sigma_x.$$

This is the induction of the K -spaces

$$\sigma_x^K \subset K \times_{NH \cap K} \sigma_x^H \subset \sigma_x.$$

The normal direction of GM^H in G_M is obtained by dividing σ_x^K

$$* \subset K \times_{NH \cap K} (\sigma_x^H/\sigma_x^K) = K \times_{NH \cap K} \nu_x^H \subset \nu_x^K.$$

Thus we conclude that the link of GM^H/G in GM^K/G is indeed $K \times_{NH \cap K} S(\nu_x)/K$. \square

2 Homotopy Property

In a stratified space, the neighborhoods of strata are often stratified systems of fibrations over the strata. The fundamental groups are related as follows.

Proposition 2.1. *Suppose $E \rightarrow X$ is a stratified system of fibrations. If the fibres are (nonempty and) connected, then $\pi_1 E \rightarrow \pi_1 X$ is surjective. If the fibres are connected and simply connected, then $\pi_1 E \rightarrow \pi_1 X$ is isomorphic.*

Proof. If $E \rightarrow X$ is a genuine fibration, then the two claims follow from the exact sequence of homotopy groups associated to the fibration.

Inductively, we only need to consider $X = Z \cup_{\partial Z} Y$, where Y is the union of lower strata, Z is the complement of a regular neighborhood of Y , and ∂Z is the boundary of the regular neighborhood as well as the boundary of Z . Correspondingly, we have $E = E_Z \cup_{E_{\partial Z}} E_Y$, such that $E_Z \rightarrow Z$ is a fibration that restricts to the fibration $E_{\partial Z} \rightarrow \partial Z$, and $E_Y \rightarrow Y$ is a stratified system of fibrations. Then we consider the map

$$\pi_1 E = \pi_1 E_Z *_{\pi_1 E_{\partial Z}} \pi_1 E_Y \rightarrow \pi_1 X = \pi_1 Z *_{\pi_1 \partial Z} \pi_1 Y.$$

If the fibres of $E \rightarrow X$ are connected, then $\pi_1 E_Z \rightarrow \pi_1 Z$ and $\pi_1 E_{\partial Z} \rightarrow \pi_1 \partial Z$ are surjective by the genuine fibration case, and $\pi_1 E_Y \rightarrow \pi_1 Y$ is surjective by induction. Therefore the map $\pi_1 E \rightarrow \pi_1 X$ is surjective. If the fibres of $E \rightarrow Z$ are connected and simply connected, then all the maps are isomorphic, so that the map $\pi_1 E \rightarrow \pi_1 X$ is isomorphic. \square

Proposition 2.2. *Suppose X is a stratified space, such that all pure strata are connected, and links are not empty, then X is connected. Moreover, if all pure strata are connected and simply connected, and all links are connected, then X is simply connected.*

We remark that a link L of a stratum X_α in another stratum X_β is stratified, with strata L_γ corresponding to the strata X_γ satisfying $X_\alpha \subsetneq X_\gamma \subset X_\beta$. Moreover, the link of L_γ in $L_{\gamma'}$ is the same as the link of X_γ in $X_{\gamma'}$. The proposition implies that, if the pure strata of the link between any two strata sandwiched between X_α and X_β are (nonempty and) connected and simply connected, then the link of X_α in X_β is simply connected.

Proof. If the links are not empty, then any pure stratum is glued to higher pure strata. Therefore the connectivity of all pure strata implies the connectivity of the union, which is the whole X .

Now assume that all pure strata are simply connected, and all links are connected. Let Y be a minimum stratum. Then we have decomposition $X = Z \cup_{\partial Z} Y$ as in the proof of Proposition 2.1. The complement Z is a stratified space, with the pure strata the same as the pure strata of X , except the stratum Y . Moreover, the links in Z are the same as the links in X . By induction, we may assume Z is simply connected. Moreover, Y is already assumed to be simply connected. The fibre of the fibration $\partial Z \rightarrow Y$ is the link of Y in X , which is assumed connected. Therefore we know ∂Z is connected. Then by Van-Kampen theorem, $\pi_1 X = \pi_1 Z *_{\pi_1 \partial Z} \pi_1 Y$ is trivial. \square

Proposition 2.3. *Suppose X is a stratified space, and Y is a closed union of strata of X . If for any link between strata of X , those pure strata of the link that are not contained in Y are connected and simply connected, then $\pi_1(X - Y) = \pi_1 X$.*

In case the stratification is $X \supset X_{-1} \supset X_{-2} \supset \cdots$, and $Y = X_{-i}$, the link L of X_{-j} in X_{-k} is stratified

$$L = L_{-k} \supset L_{-k-1} \supset \cdots \supset L_{-j+1},$$

where L_{-l} corresponds to X_{-l} . The condition of the proposition is that $L_{-l} - L_{-l-1}$ is connected and simply connected for any $k \leq l < i$.

Proof. We have decomposition $X = Z \cup_{\partial Z} Y$ as in the proof of Proposition 2.1. The fibre of the stratified system of fibrations $\partial Z \rightarrow Y$ is a stratified space L_y depending on the location of the point $y \in Y$. If Y_y is the pure stratum containing y , then the pure strata of L_y are the pure strata of the link of Y_y in X that are not contained in Y . By Proposition 2.2 and the remark afterwards, the assumption of the proposition implies that L_y is connected and simply connected. Then we may apply Proposition 2.1 to get $\pi_1 \partial Z = \pi_1 Y$. Further application of the Van-Kampen theorem gives us $\pi_1 X = \pi_1 Z *_{\pi_1 \partial Z} \pi_1 Y = \pi_1 Z = \pi_1(X - Y)$. \square

Now we turn to our multiaxial $U(n)$ -manifold M . The map $M \rightarrow \bar{M} = M/U(n)$ is a stratified system of fibrations with fibre $U(n)/U(i)$ over the pure stratum \bar{M}^{-i} . The fibre is always connected, and is simply connected when $i > 0$. Therefore by Proposition 2.1, the map $\pi_1 M_{-i} \rightarrow \pi_1 \bar{M}_{-i}$ is isomorphic for $i > 0$ and is surjective for $i = 0$.

The link of M_{-i} in M is a sphere S^d with multiaxial $U(i)$ -action. The sphere has no fixed point by $U(i)$, and the fixed points $(S^d)^{U(j)}$ by smaller unitary groups are also spheres. The orbit space $S^d/U(i)$ is the link of \bar{M}_{-i} in \bar{M} . By Proposition 1.4, the strata of $S^d/U(i)$ are the links of \bar{M}_{-i} in \bar{M}_{-j}

$$(S^d)^{U(j)}/U(i-j) = (U(i) \times_{U(i-j)} (S^d)^{U(j)})/U(i). \quad (3)$$

Proposition 2.4. *The link of \bar{M}_{-i-1} in \bar{M}_{-i} is homotopic to $\mathbb{C}P^{r_i}$, and $r_i = r_0 + i$.*

Proof. The link of M_{-1} in M is a sphere, with free action by $U(1)$, which is the circle group S^1 . Thus the sphere has to be odd dimensional, and the quotient of the sphere by the circle is homotopic to a complex projective space $\mathbb{C}P^{r_0}$. This quotient is the link of \bar{M}_{-1} in \bar{M} . By (2) and the remark afterwards, the link of \bar{M}_{-i-1} in \bar{M}_{-i} is also homotopic to a complex projective space $\mathbb{C}P^{r_i}$.

The link of M_{-2} in M is a sphere S^d with multiaxial $U(2)$ -action. The isotropy groups are either trivial or conjugate to $U(1)$. The fixed set $(S^d)^{U(1)} = S^{d'}$ has free action by the Weyl group $U(1)$ of $U(1)$ in $U(2)$. By Proposition 1.4, the quotient $S^{d'}/U(1)$ is the link of \bar{M}_{-2} in \bar{M}_{-1} . Therefore

$$2r_1 + 1 = d'. \quad (4)$$

On the other hand, the lower stratum of S^d is $U(2) \times_{U(1) \times U(1)} (S^d)^{U(1)}$ (where $U(1) \times U(1)$ is the normalizer of $U(1)$ in $U(2)$), which corresponds to M_{-1} . Since the link of M_{-1} in M is S^{2r_0+1} , we get

$$\begin{aligned} 2r_0 + 1 &= \dim S^d - \dim(U(2) \times_{U(1) \times U(1)} (S^d)^{U(1)}) - 1 \\ &= d - d' - \dim U(2) + \dim U(1) \times U(1) - 1 \\ &= d - d' - 3. \end{aligned} \quad (5)$$

By a theorem of Borel [2, page 175], we have

$$\begin{aligned} d + 1 &= \dim S^d + 1 \\ &= (\dim(S^d)^{U(1) \times 1} + 1) + (\dim(S^d)^{1 \times U(1)} + 1) \\ &= 2(\dim(S^d)^{U(1)} + 1) = 2(d' + 1). \end{aligned} \quad (6)$$

From (4), (5) and (6), we get $r_1 = r_0 + 1$. By (2), we have similar relation $r_{i+1} = r_i + 1$. \square

Proposition 2.5. *All strata and pure strata of the links in \bar{M} are connected and simply connected, and*

$$\pi_1(\bar{M}_{-i} - \bar{M}_{-j}) = \pi_1(\bar{M}_{-i}), \quad j > i.$$

In particular, the pure stratum \bar{M}^{-i} and the stratum \bar{M}_{-i} have the same fundamental group.

Proof. We first prove that the top pure stratum of the link of \bar{M}_{-i} in \bar{M} is simply connected. The link is the orbit space of the link of M_{-1} in M , which is a sphere S^d with multi-axial $U(i)$ -action. By Proposition 1.4, the next stratum of the sphere, which corresponds to M_{-1} , is $U(i)(S^d)^{U(1)} = U(i) \times_{U(i-1) \times U(1)} (S^d)^{U(1)}$. The top pure stratum of the link $S^d/U(i)$ is

$$S^d/U(i) - (S^d)^{U(1)}/U(i-1) = (S^d - U(i)(S^d)^{U(1)})/U(i).$$

We will show that this is simply connected.

Since S^d and $U(i)(S^d)^{U(1)}$ correspond to M and M_{-1} , by Proposition 2.4, we have

$$\dim S^d - \dim U(i)(S^d)^{U(1)} = \dim M - \dim M_{-1} = 2r_0 + 2.$$

If $r_0 > 0$, then the dimension gap is > 3 , so that

$$\pi_1(S^d - U(i)(S^d)^{U(1)}) = \pi_1 S^d.$$

By $d \geq 2r_0 + 2 > 3$, the complement $S^d - U(i)(S^d)^{U(1)}$ is simply connected. Since $U(i)$ is connected and acts freely on the complement, the quotient is also simply connected.

What about the case $r_0 = 0$? In this case, we consider the stratum $U(i)(S^d)^{U(2)}$ of the link S^d two steps down, which corresponds to M_{-2} . Deleting this stratum, we get a two strata $U(i)$ -space $X = S^d - U(i)(S^d)^{U(2)}$ with lower stratum

$$\begin{aligned} X_{-1} &= U(i)(S^d)^{U(1)} - U(i)(S^d)^{U(2)} \\ &= U(i) \times_{U(i-1) \times U(1)} ((S^d)^{U(1)} - U(i-1)(S^d)^{U(2)}) \\ &= U(i) \times_{U(i-1) \times U(1)} (S^{d'} - U(i-1)(S^{d'})^{U(1)}). \end{aligned}$$

The dimension gaps

$$\begin{aligned} \dim S^{d'} - \dim U(i-1)(S^{d'})^{U(1)} &= \dim M_{-1} - \dim M_{-2} = 2r_1 + 2 = 2r_0 + 4 \geq 4, \\ \dim S^d - \dim U(i)(S^d)^{U(2)} &= \dim M - \dim M_{-2} \geq \dim M_{-1} - \dim M_{-2} \geq 4. \end{aligned}$$

Therefore X and $S^{d'} - U(i-1)(S^{d'})^{U(1)}$ are simply connected. Moreover, by the fibration (and i must be > 1 in current discussion)

$$S^{d'} - U(i-1)(S^{d'})^{U(1)} \rightarrow X_{-1} \rightarrow U(i)/U(i-1),$$

we know X_{-1} is also simply connected.

We have a decomposition $X = Z \cup_{\partial Z} X_{-1}$, where Z is the top pure stratum, and ∂Z is the boundary of an equivariant regular neighborhood of X_{-1} and is also the boundary of Z . The top stratum Z has free $U(i)$ -action, and we want to show $Z/U(i)$ is simply connected.

The assumption $r_0 = 0$ means that we have a fibration

$$S^1 \rightarrow \partial Z \rightarrow X_{-1},$$

where $U(i)$ acts freely on Z , and S^1 is the orbit from the action of a subgroup conjugate to $U(1)$. Since the inclusion of S^1 in $U(i)$ induces $\pi_1 S^1 = \pi_1 U(i)$, we get

$$\pi_1(\partial Z/U(i)) = \pi_1 X_{-1} = \{1\}.$$

On the other hand, by Van-Kampen theorem, we have

$$\{1\} = \pi_1 X = \pi_1 Z *_{\pi_1 \partial Z} \pi_1 X_{-1} = \pi_1 Z *_{\pi_1 \partial Z} \{1\}.$$

This implies that $\pi_1 \partial Z \rightarrow \pi_1 Z$ is onto. Since $U(i)$ is connected, the composition $\pi_1 \partial Z \rightarrow \pi_1 Z \rightarrow \pi_1(Z/U(i))$ is also onto. On the other hand, the composition factors through $\pi_1(\partial Z/U(i))$, which we have shown to be trivial. Therefore $\pi_1(Z/U(i))$ is the trivial group.

The formula (3) shows that the other pure strata of links are comparable to the top pure strata of \bar{M}_{-i} in \bar{M} . Therefore they are also connected and simply connected. The other claims in the proposition follow from Propositions 2.2 (and the remark afterwards) and 2.3. \square

3 General Splitting Theorem

The homotopy property of multi-axial $U(n)$ -manifolds makes it possible to split its iso-variant structure set. In fact, we have splitting in the following general set up.

Theorem 3.1. *Suppose $X = X_0 \supset X_{-1} \supset X_{-2} \supset \cdots$ is a stratified space, satisfying the following properties*

1. *The link of X_{-1} in X is homotopic to $\mathbb{C}P^r$ with even r .*
2. *The link bundle of $X_{-1} - X_{-2}$ in X is orientable.*
3. *The top two pure strata of the link of X_{-i} in X are connected and simply connected.*

Then there is a natural homotopy equivalence of surgery obstructions

$$\mathbb{L}(X) = \mathbb{L}(X, \text{rel } X_{-2}) \oplus \mathbb{L}(X_{-2}).$$

Moreover, we have

$$\mathbb{L}(X, \text{rel } X_{-2}) = \mathbb{L}(\pi_1 X, \pi_1 X_{-1}).$$

The surgery obstruction $\mathbb{L}(\pi_1 X, \pi_1 X_{-1})$ in the last equality is used for the surgery problems over a manifold Z with boundary ∂Z satisfying $\dim Z = \dim X$, $\pi_1 Z = \pi_1 X$ and $\pi_1 \partial Z = \pi_1 X_{-1}$.

It will be clear from the subsequent argument that $\mathbb{C}P^r$ is used to get the periodicity for the classical surgery obstructions. Therefore it can be replaced by any manifold of signature one.

The second condition simply means that the monodromy action preserves the fundamental class of $\mathbb{C}P^r$.

In the third condition, let E be the boundary of a regular neighborhood of X_{-2} in X . Then the regular neighborhood is the mapping cylinder of a map $E \rightarrow X_{-2}$, which fits into a stratified system of fibrations

$$L \rightarrow E \rightarrow X_{-2}.$$

This is a fibration over the pure stratum $X_{-i} - X_{-i-1}$, $i \geq 2$, with the fibre L being the top two pure strata of the link of X_{-i} in X .

The boundary E is a two strata space with lower stratum $E_{-1} = E \cap X_{-1}$. So we actually have a stratified system of fibrations

$$(L, L_{-1}) \rightarrow (E, E_{-1}) \rightarrow X_{-2}, \quad L_{-1} = L \cap E_{-1} = L \cap X_{-1}.$$

The third condition means that $L - L_{-1}$ and L_{-1} are connected and simply connected. We also note that both the link of E_{-1} in E and the link of L_{-1} in L are the link of X_{-1} in X , which is $\mathbb{C}P^r$ by the first condition.

Thus the condition of Proposition 2.2 is satisfied by L , so that L is also connected and simply connected. In fact this follows directly from a simple computation by the fundamental group of fibration and Van-Kampen theorem. Moreover, the condition of Proposition 2.3 is satisfied by $Y = X_{-1}$ and $Y = X_{-2}$, so that

$$\pi_1 X = \pi_1(X - X_{-1}) = \pi_1(X - X_{-2}). \quad (7)$$

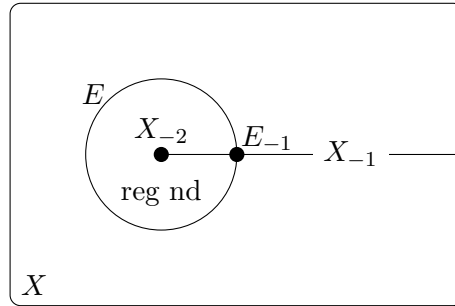


Figure 1: Regular neighborhood of X_{-2} in X

To prove the theorem, we first establish the following result, which is essentially a reformulation of the periodicity for the classical surgery obstruction [10, Theorem 9.9].

Proposition 3.2. *Suppose X is a two strata space, such that the link bundle of X_{-1} in X is an orientable fibration with fibre homotopy equivalent to $\mathbb{C}P^r$ for some even r . Then*

$$\mathbb{L}(X) = \mathbb{L}(X, \text{rel } X_{-2}) = \mathbb{L}(\pi_1 X, \pi_1 X_{-1}).$$

Proof. We have $X = Z \cup_{\partial Z} X_{-1}$, where Z is the top pure stratum, and ∂Z is the boundary of a regular neighborhood of X_{-1} and is also the boundary of Z . Moreover, the assumption says that $\partial Z \rightarrow X_{-1}$ is an orientable fibration with fibre homotopy equivalent to $\mathbb{C}P^r$.

The surgery obstruction $\mathbb{L}(X)$ of the two strata space X naturally maps to the surgery obstruction $\mathbb{L}(Z)$ of the manifold Z with boundary ∂Z . The map fits into a fibration

$$\mathbb{L}(\partial Z \times [0, 1] \cup X_{-1}) \rightarrow \mathbb{L}(X) \rightarrow \mathbb{L}(Z).$$

Here the mapping cylinder $\partial Z \times [0, 1] \cup X_{-1}$ is a regular neighborhood of X_{-1} in X and is a two strata space with X_{-1} as lower stratum. The surgery obstruction further fits into a fibration

$$\mathbb{L}(\partial Z \times [0, 1] \cup X_{-1}) \xrightarrow{\text{rest}} \mathbb{L}(X_{-1}) \xrightarrow{\text{trf}} \mathbb{L}(\partial Z),$$

given by the restriction and the transfer. Since $\partial Z \rightarrow X_{-1}$ is an orientable fibration with fibre homotopy equivalent to $\mathbb{C}P^r$ for some even r , by the classical periodicity for the surgery obstruction [8, 9], the transfer map is a homotopy equivalence. Therefore the second fibration tells us that $\mathbb{L}(\partial Z \times [0, 1] \cup X_{-1})$ is contractible, and then the first fibration tells us

$$\mathbb{L}(X) = \mathbb{L}(Z) = \mathbb{L}(\pi_1 Z, \pi_1 \partial Z).$$

The fibration $\mathbb{C}P^r \rightarrow \partial Z \rightarrow X_{-1}$ implies $\pi_1 \partial Z = \pi_1 X_{-1}$. Then by Van-Kampen theorem, $\pi_1 X = \pi_1 Z *_{\pi_1 \partial Z} \pi_1 X_{-1} = \pi_1 Z$. \square

Proof of Theorem 3.1. We adopt the notation in the discussion after the theorem. Let Z be the complement of the regular neighborhood of X_{-2} . Then Z is a two strata space with lower stratum $Z_{-1} = Z \cap X_{-1}$. On the other hand, the two strata space E is the boundary of Z in the sense that a neighborhood of E in Z is a collar. We will use Z and E to denote the spaces equipped with two strata, and use (Z, E) to denote the space Z considered as four strata space, in which the two strata of E are also counted.

The inclusion $Z \rightarrow X$ is a stratified map and induces a map $\mathbb{L}(Z) \rightarrow \mathbb{L}(X)$. Moreover, we also have a natural map $\mathbb{L}(X) \rightarrow \mathbb{L}(Z, E)$ to the four strata space (Z, E) . The composition

$$\mathbb{L}(Z) \rightarrow \mathbb{L}(X) \rightarrow \mathbb{L}(Z, E) \tag{8}$$

is the natural inclusion map that fits into a fibration

$$\mathbb{L}(Z) \rightarrow \mathbb{L}(Z, E) \rightarrow \mathbb{L}(E). \tag{9}$$

The two strata space E satisfies the condition of Proposition 3.2, so that $\mathbb{L}(E) = \mathbb{L}(\pi_1 E, \pi_1 E_{-1})$. We have stratified systems of fibrations

$$L \rightarrow E \rightarrow X_{-2}, \quad L_{-1} \rightarrow E_{-1} \rightarrow X_{-2}.$$

Since L and L_{-1} are connected and simply connected, by Proposition 2.1, we have

$$\pi_1 E = \pi_1 E_{-1} = \pi_1 X_{-2}. \tag{10}$$

Then it follows from the π - π theorem of the classical surgery theory that $\mathbb{L}(E)$ is contractible. Therefore by the fibration (9), the natural map $\mathbb{L}(Z) \rightarrow \mathbb{L}(Z, E)$ is a homotopy equivalence. This further implies that in the composition (8), the first map is naturally split injective, and the second is naturally split surjective. On the other hand, the first map fits into another fibration

$$\mathbb{L}(Z) = \mathbb{L}(X, \text{rel } X_{-2}) \rightarrow \mathbb{L}(X) \rightarrow \mathbb{L}(X_{-2}).$$

Therefore the natural splitting induces a natural homotopy equivalence

$$\mathbb{L}(X) = \mathbb{L}(Z) \oplus \mathbb{L}(X_{-2}) = \mathbb{L}(X, \text{rel } X_{-2}) \oplus \mathbb{L}(X_{-2}).$$

Proposition 3.2 can be applied to Z and gives us $\mathbb{L}(Z) = \mathbb{L}(\pi_1 Z, \pi_1 Z_{-1})$. By (7), we have $\pi_1 Z = \pi_1(X - X_{-2}) = \pi_1 X$. By (10) and Van-Kampen theorem, we have $\pi_1 Z_{-1} = \pi_1 X_{-1}$. \square

The natural splitting of the surgery obstruction induces a natural splitting of the structure set. The next section proves the special case of multiaxial $U(n)$ -manifolds, and that proof contains all the key ingredients for proving the general result. Therefore we simply present the general result without giving proof.

Theorem 3.3. *Suppose $X = X_0 \supset X_{-1} \supset X_{-2} \supset \cdots$ is a stratified space, satisfying the following properties*

1. *The link of X_{-1} in X is homotopic to $\mathbb{C}P^r$ with even r .*
2. *The link bundle of $X_{-1} - X_{-2}$ in X is orientable.*
3. *The pure strata of all links are connected and simply connected.*

Then there is a natural homotopy equivalence of structure sets

$$\mathbb{S}(X) = \mathbb{S}(X, \text{rel } X_{-2}) \oplus \mathbb{S}(X_{-2}).$$

Moreover, $\mathbb{S}(X, \text{rel } X_{-2})$ is the homotopy fibre of the assembly map

$$\mathbb{H}(X, X_{-1}; \mathbb{L}) \rightarrow \mathbb{L}(\pi_1 X, \pi_1 X_{-1}).$$

We remark that $\mathbb{C}P^r$ can be replaced by any manifold of signature one.

4 Structure Set of Multiaxial Action

By Proposition 2.5, the pure strata of links in a multiaxial $U(n)$ -manifold are all simply connected. Then by the topological h -cobordism theory [1, 7], the neighborhoods of strata have block bundle structure, and the manifold can be considered as being of the PT category [11]. By the stratified surgery theory in [11], the structure set $\mathbb{S}_{U(n)}(M)$ can be computed by the “unstable surgery fibration”

$$\mathbb{S}_{U(n)}(M) \rightarrow \mathbb{H}(\bar{M}; \mathbb{L}_{U(n)}(\text{loc } M)) \rightarrow \mathbb{L}_{U(n)}(M). \quad (11)$$

From the discussion in Section 2, if r_0 is even, then the orbit space \bar{M} satisfies the conditions of Theorem 3.1. Therefore we have the natural splitting

$$\mathbb{L}_{U(n)}(M) = \mathbb{L}_{U(n)}(M, \text{rel } M_{-2}) \oplus \mathbb{L}_{U(n)}(M_{-2}) = \mathbb{L}(\pi_1 \bar{M}, \pi_1 \bar{M}_{-1}) \oplus \mathbb{L}(\bar{M}_{-2}).$$

By (2), the surgery obstruction of \bar{M}_{-2} is the same as the surgery obstruction of the multiaxial $U(n-2)$ -manifold $M^{U(2)}$. Moreover, by Proposition 2.4, r_0 even implies $r_{-2} =$

$r_0 + 2$ is also even. Therefore the surgery obstruction of \bar{M}_{-2} can be further naturally split

$$\begin{aligned}\mathbb{L}_{U(n)}(M_{-2}) &= \mathbb{L}_{U(n-2)}(M^{U(2)}) \\ &= \mathbb{L}_{U(n-2)}(M^{U(2)}, \text{rel } M_{-2}^{U(2)}) \oplus \mathbb{L}_{U(n-2)}(M_{-2}^{U(2)}) \\ &= \mathbb{L}_{U(n)}(M_{-2}, \text{rel } M_{-4}) \oplus \mathbb{L}_{U(n)}(M_{-4}).\end{aligned}$$

Keep going, we get the following natural decomposition.

Proposition 4.1. *Suppose M is a multiaxial $U(n)$ -manifold, such that the link of \bar{M}_{-1} in \bar{M} is $\mathbb{C}P^r$ with even r . Then we have natural splitting*

$$\mathbb{L}_{U(n)}(M) = \oplus_{i \geq 0} \mathbb{L}_{U(n)}(M_{-2i}, \text{rel } M_{-2i-2}) = \oplus_{i \geq 0} \mathbb{L}(\pi_1 \bar{M}_{-2i}, \pi_1 \bar{M}_{-2i-1}).$$

The condition for r even is equivalent to $\dim M - \dim M^{U(1)} - 2n$ is 2 mod 4. Applying the natural splitting to the surgery fibration (11), the assembly map is naturally the sum of the following assembly maps

$$\mathbb{H}(\bar{M}; \mathbb{L}_{U(n)}(\text{loc}(M_{-2i}, \text{rel } M_{-2i-2}))) \rightarrow \mathbb{L}_{U(n)}(M_{-2i}, \text{rel } M_{-2i-2}).$$

Since the coefficient of the homology is concentrated on the stratum M_{-2i} , the assembly map is actually

$$\mathbb{H}(\bar{M}_{-2i}; \mathbb{L}_{U(n)}(\text{loc}(M_{-2i}, \text{rel } M_{-2i-2}))) \rightarrow \mathbb{L}_{U(n)}(M_{-2i}, \text{rel } M_{-2i-2}).$$

By applying the stratified surgery theory to M_{-2i} relative to the lower stratum M_{-2i-2} , the homotopy fibre of the map above is actually the structure set $\mathbb{S}_{U(n-2i)}(M^{U(2i)}, \text{rel } M_{-2}^{U(2i)})$. Therefore

$$\begin{aligned}\mathbb{S}_{U(n)}(M) &= \oplus_{i \geq 0} \mathbb{S}_{U(n-2i)}(M^{U(2i)}, \text{rel } M_{-2}^{U(2i)}) \\ &= \oplus_{i \geq 0} \mathbb{S}_{U(n)}(M_{-2i}, \text{rel } M_{-2i-2}) \\ &= \oplus_{i \geq 0} \mathbb{S}(\bar{M}_{-2i}, \text{rel } \bar{M}_{-2i-2}).\end{aligned}\tag{12}$$

The top piece $\mathbb{S}_{U(n)}(M, \text{rel } M_{-2})$ in the decomposition is the homotopy fibre of the assembly map

$$\mathbb{H}(\bar{M}; \mathbb{L}_{U(n)}(\text{loc}(M, \text{rel } M_{-2}))) \rightarrow \mathbb{L}_{U(n)}(M, \text{rel } M_{-2}).$$

By Theorem 3.1, we have $\mathbb{L}_{U(n)}(M, \text{rel } M_{-2}) = \mathbb{L}(\pi_1 \bar{M}, \pi_1 \bar{M}_{-1})$. On the other hand, the coefficient $\mathbb{L}_{U(n)}(\text{loc}(M, \text{rel } M_{-2}))$ in the homology depends on the location.

1. At a point in $\bar{M} - \bar{M}_{-1}$, the coefficient is $\mathbb{L}(D^p) = \mathbb{L}(e)$, where D^p is a ball in the manifold $\bar{M} - \bar{M}_{-1}$.
2. At a point in $\bar{M}_{-1} - \bar{M}_{-2}$, the coefficient is $\mathbb{L}(c\mathbb{C}P^r \times D^p)$, where $c\mathbb{C}P^r$ is cone on the link of \bar{M}_{-1} in \bar{M} , and D^p is a ball in the manifold $\bar{M}_{-1} - \bar{M}_{-2}$. Since r is even, the surgery obstruction $\mathbb{L}(c\mathbb{C}P^r \times D^p)$ is contractible by the classical periodicity. See Proposition 3.2.

3. At a point in \bar{M}_{-2} , the point belongs to some $\bar{M}_{-i} - \bar{M}_{-i-1}$, and the coefficient is $\mathbb{L}(L \times D^p)$, where L is the top two pure strata of the link of \bar{M}_{-i} in \bar{M} , and D^p is a ball in the manifold $\bar{M}_{-i} - \bar{M}_{-i-1}$. See the discussion after Theorem 3.1. The situation fits into Proposition 3.2, and by the connectivity and simple connectivity of L and L_{-1} , the surgery obstruction $\mathbb{L}(L \times D^p) = \mathbb{L}(\pi_1 L, \pi_1 L_{-1}) = \mathbb{L}(e, e)$ is contractible.

Thus the coefficient is the surgery obstruction spectrum $\mathbb{L} = \mathbb{L}(e)$ on the top pure stratum $\bar{M} - \bar{M}_{-1}$ and is trivial on \bar{M}_{-1} . Therefore the assembly map is simply

$$\mathbb{H}(\bar{M}, \bar{M}_{-1}; \mathbb{L}) \rightarrow \mathbb{L}(\pi_1 \bar{M}, \pi_1 \bar{M}_{-1}).$$

If \bar{M} were a manifold with boundary \bar{M}_{-1} , then the homotopy fibre of the assembly map above would be the structure of the pair. Here the homotopy fibre is an algebraic computation that cannot be strictly interpreted as the structure of the pair. Therefore we denote the homotopy fibre by $\mathbb{S}^{\text{alg}}(\bar{M}, \bar{M}_{-1})$, and we have $\mathbb{S}_{U(n)}(M, \text{rel } M_{-2}) = \mathbb{S}^{\text{alg}}(\bar{M}, \bar{M}_{-1})$. The other pieces in (12) are similar, and we have

$$\mathbb{S}(\bar{M}_{-2i}, \text{rel } \bar{M}_{-2i-2}) = \mathbb{S}^{\text{alg}}(\bar{M}_{-2i}, \bar{M}_{-2i-1}).$$

We summarize the discussion into the following decomposition.

Theorem 4.2. *Suppose M is a multiaxial $U(n)$ -manifold, such that the link of \bar{M}_{-1} in \bar{M} is $\mathbb{C}P^r$ with even r . Then we have natural splitting*

$$\mathbb{S}_{U(n)}(M) = \oplus_{i \geq 0} \mathbb{S}^{\text{alg}}(\bar{M}_{-2i}, \bar{M}_{-2i-1}).$$

The evenness of r is critical for the splitting argument to carry through. In case $r = r_0$ is odd, we may not have such a splitting in general. However, if $M = W^{U(1)}$ for a multiaxial $U(n+1)$ -manifold W , then by (2), we have

$$\bar{M}_{-i} = \bar{W}_{-i-1}, \quad \mathbb{L}_{U(n)}(M_{-i}) = \mathbb{L}_{U(n+1)}(W_{-i-1}), \quad i \leq n.$$

The link of $\bar{W}_{-1} = \bar{M}$ in \bar{W} is $\mathbb{C}P^{r-1}$, and $r-1$ is even. Therefore the restriction $\mathbb{L}_{U(n+1)}(W) \rightarrow \mathbb{L}_{U(n+1)}(W_{-2})$ of the surgery obstruction naturally splits. Moreover, this restriction is a composition of two restrictions

$$\mathbb{L}_{U(n+1)}(W) \rightarrow \mathbb{L}_{U(n+1)}(W_{-1}) = \mathbb{L}_{U(n)}(M) \rightarrow \mathbb{L}_{U(n+1)}(W_{-2}) = \mathbb{L}_{U(n)}(M_{-1}),$$

so that the restriction $\mathbb{L}_{U(n)}(M) \rightarrow \mathbb{L}_{U(n)}(M_{-1})$ also naturally splits. Combined with the fibration

$$\mathbb{L}_{U(n)}(M, \text{rel } M_{-1}) \rightarrow \mathbb{L}_{U(n)}(M) \rightarrow \mathbb{L}_{U(n)}(M_{-1}),$$

we get natural splitting

$$\mathbb{L}_{U(n)}(M) = \mathbb{L}_{U(n)}(M, \text{rel } M_{-1}) \oplus \mathbb{L}_{U(n)}(M_{-1}).$$

We also note that

$$\mathbb{L}_{U(n)}(M, \text{rel } M_{-1}) = \mathbb{L}(\pi_1(\bar{M} - \bar{M}_{-1})) = \mathbb{L}(\pi_1 \bar{M}).$$

Now the multiaxial $U(n-1)$ -manifold $M^{U(1)}$ satisfies the condition of Proposition 4.1, so that further splittings of the surgery obstruction can be carried out.

Proposition 4.3. *Suppose M is a multiaxial $U(n)$ -manifold, such that the link of \bar{M}_{-1} in \bar{M} is $\mathbb{C}P^r$ with odd r . If $M = W^{U(1)}$ for a multiaxial $U(n+1)$ -manifold W , then we have natural splitting*

$$\begin{aligned}\mathbb{L}_{U(n)}(M) &= \mathbb{L}_{U(n)}(M, \text{rel } M_{-1}) \oplus \left(\oplus_{i \geq 0} \mathbb{L}_{U(n)}(M_{-2i-1}, \text{rel } M_{-2i-3}) \right) \\ &= \mathbb{L}(\pi_1 \bar{M}) \oplus \left(\oplus_{i \geq 0} \mathbb{L}(\pi_1 \bar{M}_{-2i-1}, \pi_1 \bar{M}_{-2i-2}) \right).\end{aligned}$$

The condition for r odd is equivalent to $\dim M - \dim M^{U(1)} - 2n$ is a multiple of 4. A similar argument can be made for the structure set and gives us the following decomposition.

Theorem 4.4. *Suppose M is a multiaxial $U(n)$ -manifold, such that the link of \bar{M}_{-1} in \bar{M} is $\mathbb{C}P^r$ with odd r . If $M = W^{U(1)}$ for a multiaxial $U(n+1)$ -manifold W , then we have natural splitting*

$$\mathbb{S}_{U(n)}(M) = \mathbb{S}^{\text{alg}}(\bar{M}) \oplus \left(\oplus_{i \geq 0} \mathbb{S}^{\text{alg}}(\bar{M}_{-2i-1}, \bar{M}_{-2i-2}) \right).$$

The only thing that needs to be explained is the top piece $\mathbb{S}_{U(n)}(M, \text{rel } M_{-1})$, computed as the homotopy fibre of the assembly map

$$\mathbb{H}(\bar{M}; \mathbb{L}_{U(n)}(\text{loc}(M, \text{rel } M_{-1}))) \rightarrow \mathbb{L}_{U(n)}(M, \text{rel } M_{-1}).$$

We already know $\mathbb{L}_{U(n)}(M, \text{rel } M_{-1}) = \mathbb{L}(\pi_1 \bar{M})$. Applying this to the coefficient of the homology, we know $\mathbb{L}_{U(n)}(\text{loc}(M, \text{rel } M_{-1})) = \mathbb{L}(\pi)$, where π is the fundamental group of the orbit space of the local piece of M . Since the orbit space is locally contractible, we have $\pi = e$, so that the coefficient is the usual surgery obstruction spectrum. We conclude that the assembly map is actually

$$\mathbb{H}(\bar{M}; \mathbb{L}) \rightarrow \mathbb{L}(\pi_1 \bar{M}).$$

The homotopy fibre of the map is what we denoted as $\mathbb{S}^{\text{alg}}(\bar{M})$.

5 Structure Set of Multiaxial Representation Sphere

Let ρ_n be the defining representation of $U(n)$, and let ϵ be the real 1-dimensional trivial representation. Then for any natural number k , the unit sphere

$$M = S(k\rho_n \oplus j\epsilon) = S(k\rho_n) * S^{j-1}$$

of the representation $k\rho_n \oplus j\epsilon$ is a multiaxial $U(n)$ -manifold. We want to compute the structure set of this multiaxial representation sphere.

If $k < n$, then $M = U(n) \times_{U(k)} S(k\rho_k \oplus j\epsilon)$, and the study of the $U(n)$ -manifold is the same as the study of the $U(k)$ -manifold $S(k\rho_k \oplus j\epsilon)$. Therefore without loss of generality, we will always assume $k \geq n$ in the subsequent discussion, and M is a multiaxial $U(n)$ -manifold according to our working definition.

The fixed points of M by $U(i)$ is

$$M^{U(i)} = S(k\rho_n^{U(i)} \oplus j\epsilon) = S(k\rho_{n-i} \oplus j\epsilon) = S(k\rho_{n-i}) * S^{j-1}.$$

This is a multiaxial $U(n-i)$ -representation sphere and has dimension

$$\dim M^{U(i)} = 2k(n-i) - 1 + j.$$

The dimension gap between the top two strata is

$$\begin{aligned} & \dim M - \dim U(n) \times_{U(n-1) \times U(1)} M^{U(1)} \\ &= [2kn - 1 + j] - [2k(n-1) - 1 + j + n^2 - (n-1)^2 - 1^2] \\ &= 2(k-n) + 2. \end{aligned}$$

Therefore the link of \bar{M}_{-1} in \bar{M} is $\mathbb{C}P^{k-n}$. If $k-n$ is even, then we can use Theorem 4.2 to compute the structure set. If $k-n$ is odd, then we note that $M = W^{U(1)}$ for the multiaxial $U(n+1)$ -representation sphere $W = S(k\rho_{n+1} \oplus j\epsilon)$, so that Theorem 4.4 can be used.

Now assume $k-n$ is even. We compute the top piece

$$\mathbb{S}^{\text{alg}}(\bar{M}, \bar{M}_{-1}) = \mathbb{S}^{\text{alg}}(S(k\rho_n \oplus j\epsilon)/U(n), S(k\rho_{n-1} \oplus j\epsilon)/U(n-1))$$

in the decomposition of $\mathbb{S}(\bar{M}) = \mathbb{S}_{U(n)}(S(k\rho_n \oplus j\epsilon))$. The other pieces are similar, simply by replacing n with $n-2i$.

The representation sphere is the link of the origin in the representation space $k\rho_n \oplus j\epsilon = \mathbb{C}^{kn} \oplus \mathbb{R}^j$. By Propositions 2.5, both \bar{M} and \bar{M}_{-1} are connected and simply connected. In case the action is not free, we have $\bar{M}_{-1} \neq \emptyset$, and the surgery obstruction $\mathbb{L}(\pi_1 \bar{M}, \pi_1 \bar{M}_{-1}) = \mathbb{L}(e, e) = 0$, so that the top piece is the same as the homology

$$\begin{aligned} \mathbb{H}(\bar{M}, \bar{M}_{-1}; \mathbb{L}) &= \mathbb{H}(S(k\rho_n)/U(n) * S^{j-1}, S(k\rho_{n-1})/U(n-1) * S^{j-1}; \mathbb{L}) \\ &= \mathbb{H}_{-j}(S(k\rho_n)/U(n), S(k\rho_{n-1})/U(n-1); \mathbb{L}). \end{aligned}$$

Denote by

$$N = S(k\rho_n)$$

the multiaxial representation sphere without the trivial summand. By $\dim \bar{M} = \dim M - n^2 = 2kn - n^2 - 1 + j$, we get

$$\begin{aligned} \mathbb{S}^{\text{alg}}(\bar{M}, \bar{M}_{-1}) &= \pi_{2kn-n^2-1+j} \mathbb{S}^{\text{alg}}(\bar{M}, \bar{M}_{-1}) \\ &= \pi_{2kn-n^2-1+j} \mathbb{H}(\bar{M}, \bar{M}_{-1}; \mathbb{L}) = H_{2kn-n^2-1}(\bar{N}, \bar{N}_{-1}; \mathbb{L}). \end{aligned}$$

We remark that the $U(n)$ -action is free only when $n=1$ and $j=0$. In this case, the surgery obstruction $\mathbb{L}(\pi_1 \bar{M}, \pi_1 \bar{M}_{-1}) = \mathbb{L}(\pi_1 \bar{M}) = \mathbb{L}(e)$. The case is exactly the fake complex projective space studied in Section 14C of [10].

According to a computation of Jared Bass, \bar{N} has a CW-structure that can be described as follows. An element in $S(k\rho_n)$ is a k -tuple of vectors $\xi = (v_1, v_2, \dots, v_k)$ in \mathbb{C}^k of total unit length. By writing v_i as columns, the k -tuple ξ can be expressed as a complex $k \times n$ -matrix. The action by a unitary matrix $A \in U(n)$ is simply the left multiplication $A\xi = (Av_1, Av_2, \dots, Av_k)$. The orbit space \bar{N} consists of the equivalence classes of the k -tuples under the $U(n)$ -action. Moreover, $\bar{N} - \bar{N}_{-1}$ consists of those k -tuples of full rank n , and \bar{N}_{-1} consists of those k -tuples of less than the full rank.

By using the Gram-Schmit process, up to unitary transformations, a k -tuple ξ has a unique representative of the form

$$\bar{\xi} = \begin{pmatrix} \lambda_1 & \cdots & * & \cdots & * & \cdots & * & \cdots \\ & & \lambda_2 & \cdots & * & \cdots & * & \cdots \\ & & & & \lambda_3 & \cdots & * & \cdots \\ & & & & & \ddots & \vdots & \cdots \\ & & & & & & \lambda_d & \cdots \end{pmatrix},$$

where the empty spaces are occupied by 0, $*$ and dots mean complex numbers, $\lambda_i > 0$, and the total length of all the entries is 1. Let λ_j appear m_j place from the right end of the matrix (i.e., λ_j lies in the $k - m_j + 1$ column), and call (m_1, m_2, \dots, m_d) the *shape* of the matrix. Then for any $k \geq m_1 > m_2 > \cdots > m_d > 0$, $d \leq n$, all the matrices of the shape (m_1, m_2, \dots, m_d) form a cell, still denoted by (m_1, m_2, \dots, m_d) . The cell has dimension

$$\dim(m_1, m_2, \dots, m_d) = 2(m_1 + m_2 + \cdots + m_d) - d - 1.$$

Geometrically, the boundary of the cell (m_1, m_2, \dots, m_d) consists of those shapes $(m'_1, m'_2, \dots, m'_{d'})$ satisfying $d' \leq d$ and $m'_i \leq m_i$, with at least one inequality to be strict. In homological computation, only those shapes of one dimension less matter. This means that

$$m_d = 1, \quad d' = d - 1, \quad m'_i = m_i \text{ for } 1 \leq i < d.$$

Therefore the only nontrivial boundary map of the cellular chain complex is

$$\partial(m_1, m_2, \dots, m_{d-1}, 1) = (m_1, m_2, \dots, m_{d-1}).$$

The homology is then freely generated by the shapes that are neither $(m_1, m_2, \dots, m_{d-1}, 1)$ nor $(m_1, m_2, \dots, m_{d-1})$ in the equality above. These are exactly the shapes satisfying $d = n$ and $m_n > 1$, and the shape (1) (meaning $d = 1$ and $m_1 = 1$). The shape (1) is the base point of \bar{N} .

The homology $H_*(\bar{N}, \bar{N}_{-1}; \mathbb{L})$ is the limit of a spectral sequence with

$$E_2^{p,q} = H_p(\bar{N}, \bar{N}_{-1}; \pi_q \mathbb{L}) = \begin{cases} H_p(\bar{N}, \bar{N}_{-1}; \mathbb{Z}), & \text{if } q \equiv 0 \pmod{4}, \\ H_p(\bar{N}, \bar{N}_{-1}; \mathbb{Z}_2), & \text{if } q \equiv 2 \pmod{4}, \\ 0, & \text{if } q \text{ is odd.} \end{cases}$$

These can be computed by using the cellular chain complex of Jared Bass. Note that the condition $d = n$ and $m_n > 1$ implies that the cells have full rank and therefore lie in the top stratum of \bar{N} . The condition also implies that the dimensions of the cells have the same parity as $n + 1$ and have range

$$n^2 + 2n - 1 \leq \dim(m_1, m_2, \dots, m_n) = 2(m_1 + m_2 + \cdots + m_n) - n - 1 \leq 2kn - n^2 - 1.$$

Incidentally, if $k = n$, the inequality cannot hold and the homology is trivial.

The same parity property implies that $E_2^{p,q}$ already collapses and

$$H_{2kn-n^2-1}(\bar{N}, \bar{N}_{-1}; \mathbb{L}) = \oplus_{p+q=2kn-n^2-1} E_2^{p,q}.$$

We have

$$E_2^{p,4s} = H_{2kn-n^2-1-4s}(\bar{N}, \bar{N}_{-1}; \mathbb{Z}) = \mathbb{Z}^{a_s},$$

where a_s is the number of shapes (m_1, m_2, \dots, m_n) satisfying

$$m_n > 1, \quad 2(m_1 + m_2 + \dots + m_n) - n - 1 = 2kn - n^2 - 1 - 4s.$$

Let $\mu_i = m_i - (n + 2 - i)$. Then we are counting the n -tuples $(\mu_1, \mu_2, \dots, \mu_n)$ satisfying

$$k - n - 1 \geq \mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq 0, \quad \mu_1 + \mu_2 + \dots + \mu_n = n(k - n - 1) - 2s.$$

Note that the first condition already implies $0 \leq \sum \mu_i \leq n(k - n - 1)$. Therefore the range for s is $0 \leq 2s \leq n(k - n - 1)$. This also implies $p = 2kn - n^2 - 1 - 4s \geq 0$. Therefore

$$\oplus_{p+4s=2kn-n^2-1} E_2^{p,4s} = \mathbb{Z}^{\sum_{0 \leq 2s \leq n(k-n-1)} a_s}.$$

The sum $\sum_{0 \leq 2s \leq n(k-n-1)} a_s$ is the number of n -tuples $(\mu_1, \mu_2, \dots, \mu_n)$ satisfying

$$k - n - 1 \geq \mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq 0, \quad \sum \mu_i + n \text{ is even.}$$

Here the parity condition is due to the assumption that $k - n$ is even. The sum of those $E_2^{p,q}$ with $q = 2 \bmod 4$ can be similarly computed.

Proposition 5.1. $H_{2kn-n^2-1}(\bar{N}, \bar{N}_{-1}; \mathbb{L}) = \mathbb{Z}^{A_{k,n}} \oplus \mathbb{Z}^{B_{k,n}}$, where $A_{k,n}$ is the number of n -tuples $(\mu_1, \mu_2, \dots, \mu_n)$ satisfying

$$k - n - 1 \geq \mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq 0, \quad \sum \mu_i + n \text{ is even,}$$

and $B_{k,n}$ is the number of n -tuples satisfying

$$k - n - 1 \geq \mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq 0, \quad \sum \mu_i + n \text{ is odd,}$$

We remark that $A_{k,n} + B_{k,n}$ is the number of n -tuples $(\mu_1, \mu_2, \dots, \mu_n)$ satisfying

$$k - n - 1 \geq \mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq 0.$$

The number is

$$A_{k,n} + B_{k,n} = \binom{k-1}{n}.$$

Note that $\binom{k-1}{n} = 0$ when $k = n$.

The other pieces in the decomposition for the structure set $S_{U(n)}(S(k\rho_n \oplus j\epsilon))$ can be similarly computed and are equal to the homology in the proposition, after substituting n by $n - 2i$. However, in two cases, the last piece in the decomposition is of different type.

The first exception is the case n is even and $j > 0$. By taking $2i = n$, the last piece is $S^{\text{alg}}(\bar{M}_{-n}) = S(S(j\epsilon))$. (Here the first S in $S(S(j\epsilon))$ means the structure set, not sphere.) Since the structure set of the sphere $S(j\epsilon) = S^{j-1}$ is trivial, this means that we only consider pieces with $2i < n$.

The second exception is the case n is odd and $j = 0$. By taking $2i + 1 = n$, the last piece is $S^{\text{alg}}(\bar{M}_{-n-1}) = S(\mathbb{C}P^{k-1})$. (Here S in $S(\mathbb{C}P^{k-1})$ means the structure set, not

sphere.) The homology is still given by the proposition, but the surgery obstruction is \mathbb{Z} instead of 0. This reduces the number of copies of \mathbb{Z} by 1 in the computation of the structure set.

In summary, we have the following computation of the structure set of the multi-axial $U(n)$ -representation sphere in half of the cases.

Theorem 5.2. *If $k \geq n$ and $k - n$ is even, then*

$$S_{U(n)}(S(k\rho_n \oplus j\epsilon)) = \mathbb{Z}^A \oplus \mathbb{Z}_2^B,$$

where

$$A = \sum_{0 \leq 2i < n} A_{k,n-2i}, \quad B = \sum_{0 \leq 2i < n} B_{k,n-2i}$$

for the $A_{k,n}$ and $B_{k,n}$ given in Proposition 5.1. The formula holds in general, with the only exception that, in case $j = 0$ and n is odd, A should be replaced by $A - 1$.

A remark on the notation: In $S_{U(n)}(S(k\rho_n \oplus j\epsilon))$, the first S means the isovariant structure set, and second S means the unit sphere.

Next we turn to the case $k - n$ is odd. The only thing we need to discuss is the top piece $\mathbb{S}^{\text{alg}}(\bar{M})$, computed by the surgery fibration

$$\mathbb{S}^{\text{alg}}(\bar{M}) \rightarrow \mathbb{H}(\bar{M}; \mathbb{L}) \rightarrow \mathbb{L}(\pi_1 \bar{M}).$$

Since \bar{M} is simply connected, $\mathbb{L}(\pi_1 \bar{M})$ is the usual surgery spectrum \mathbb{L} , and the assembly map is simply induced by the map from \bar{M} to the single point. Therefore

$$S^{\text{alg}}(\bar{M}) = \tilde{H}_{2kn-n^2-1+j}(\bar{M}; \mathbb{L})$$

is the reduced homology.

If $j > 0$, then $\tilde{H}_{2kn-n^2-1+j}(\bar{M}; \mathbb{L}) = H_{2kn-n^2-1}(\bar{N}; \mathbb{L})$. Since the cells of \bar{N} that gives nontrivial elements in the homology are either concentrated in the top stratum or is the base point given by the shape (1), the homology $H_{2kn-n^2-1}(\bar{N}; \mathbb{L})$ is still computed by Proposition 5.1, plus the homology at the base

$$H_0(\bar{N}; \pi_{2kn-n^2-1}\mathbb{L}) = L_{2kn-n^2-1}(e) = \begin{cases} \mathbb{Z}_2, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

If $j = 0$, then $\tilde{H}_{2kn-n^2-1+j}(\bar{M}; \mathbb{L}) = \tilde{H}_{2kn-n^2-1}(\bar{N}; \mathbb{L})$. This is the homology as computed by Proposition 5.1, where the homology at the base point is excluded.

We also need to consider the case the last piece in the decomposition for $S_{U(n)}(S(k\rho_n \oplus j\epsilon))$ is $S(\mathbb{C}P^{k-1})$, similar to the exceptional case in Theorem 5.2. This happens when n is even, $j = 0$ and $2i + 1 = n - 1$. In this case, the number of copies of \mathbb{Z} should be reduced by 1.

Theorem 5.3. *If $k \geq n$ and $k - n$ is odd, then*

$$S_{U(n)}(S(k\rho_n \oplus j\epsilon)) = \mathbb{Z}^A \oplus \mathbb{Z}_2^B,$$

where

$$A = A_{k,n} + \sum_{0 \leq 2i+1 < n} A_{k,n-2i-1}, \quad B = B_{k,n} + \sum_{0 \leq 2i+1 < n} B_{k,n-2i-1}$$

for the $A_{k,n}$ and $B_{k,n}$ given in Proposition 5.1. The formula holds in general, with the following exceptions

1. If $j = 0$ and n is even, then A should be replaced by $A - 1$.
2. If $j > 0$ and n is odd, then B should be replaced by $B + 1$.

6 Multiaxial $Sp(n)$ -manifold

The symplectic group $Sp(n)$ consists of $n \times n$ quaternionic matrices that preserve the standard hermitian form on \mathbb{H}^n

$$\langle x, y \rangle = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n.$$

We define a multiaxial $Sp(n)$ -manifold as a manifold with locally linear $Sp(n)$ action, such that any isotropy group is conjugate to a symplectic subgroup $Sp(i)$, and the action has free points. As illustrated by the discussion in [4], all our discussion about multiaxial $U(n)$ -manifolds is still valid, with $U(1) = S^1$ replaced by $Sp(1) = S^3$ and $\mathbb{C}P^r$ replaced by $\mathbb{H}P^r$. Here are some key points.

1. If S^3 (the group of quaternions of unit length) acts freely on a sphere, then the dimension of the sphere is $3 \bmod 4$, and the quotient is $\mathbb{H}P^r$.
2. Borel's theorem is about the dimension of fixed points of tori. The formula (6) still holds because $M^{Sp(i)} = M^{T^i}$ and all tori in $Sp(n)$ of the same dimension are conjugate.
3. $\mathbb{H}P^r$ is a manifold of signature one for even r . So results in Section 3 still hold after replacing $\mathbb{C}P^r$ by $\mathbb{H}P^r$.
4. The unique representative of k -tuples by Jared Bass up to the $U(n)$ -action is a consequence of the fact that $GL(n, \mathbb{C}) = U(n)N$, where $U(n)$ is the maximal compact subgroup of the semisimple Lie group $SL(n, \mathbb{C})$ and N is the upper triangular matrix with positive diagonal entries. This is a special example of the Iwasawa decomposition. When the decomposition is applied to the semisimple Lie group $SL(n, \mathbb{H})$, for which $Sp(n)$ is the maximal compact subgroup, we get $GL(n, \mathbb{H}) = Sp(n)N$. Therefore the description by Jared Bass is still valid, except

$$\dim(m_1, m_2, \dots, m_d) = 4(m_1 + m_2 + \cdots + m_d) - 3d - 1.$$

In what follows, we outline the computation of the structure of the representation sphere

$$M = S(k\rho_n \oplus j\epsilon) = S(k\rho_n) * S^{j-1},$$

where $\rho_n = \mathbb{H}^n$ is the defining representation of $Sp(n)$, and ϵ is the real 1-dimensional trivial representation. We have

$$\dim M = 4kn - 1 + j, \quad \dim \bar{M} = 4kn - n(2n + 1) - 1 + j.$$

If $k - n$ is even, then the top piece in the decomposition of the structure set of M is

$$\begin{aligned} S^{\text{alg}}(\bar{M}, \bar{M}_{-1}) &= \pi_{4kn - n(2n+1) - 1 + j} \mathbb{H}(\bar{M}, \bar{M}_{-1}; \mathbb{L}) \\ &= H_{4kn - n(2n+1) - 1}(\bar{N}, \bar{N}_{-1}; \mathbb{L}) \\ &= \oplus_{p+q=4kn - n(2n+1) - 1} H_p(\bar{N}, \bar{N}_{-1}; \pi_q \mathbb{L}). \end{aligned}$$

For $q = 4s$, we have

$$E_2^{p,4s} = H_{4kn-n(2n+1)-1-4s}(\bar{N}, \bar{N}_{-1}; \mathbb{Z}) = \mathbb{Z}^{a_s},$$

where a_s is the number of shapes (m_1, m_2, \dots, m_n) satisfying

$$m_n > 1, \quad 4(m_1 + m_2 + \dots + m_n) - 3n - 1 = 4kn - n(2n + 1) - 1 - 4s.$$

The condition is the same as

$$m_n > 1, \quad m_1 + m_2 + \dots + m_n = kn - \frac{1}{2}n(n - 1) - s.$$

Let $\mu_i = m_i - (n + 2 - i)$. Then we are counting the n -tuples $(\mu_1, \mu_2, \dots, \mu_n)$ satisfying

$$k - n - 1 \geq \mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq 0, \quad \mu_1 + \mu_2 + \dots + \mu_n = n(k - n - 1) - s.$$

For $q = 4s + 2$, we find the condition for (m_1, m_2, \dots, m_n) is contradictory in parity. Therefore for $Sp(n)$ -representation sphere, the computation corresponding to Proposition 5.1 is the following.

Proposition 6.1. $H_{4kn-n(2n+1)-1}(\bar{N}, \bar{N}_{-1}; \mathbb{L}) = \mathbb{Z}^{\binom{k-1}{n}}.$

This leads to the following computations of the structure set.

Theorem 6.2. *If $k \geq n$ and $k - n$ is even, then*

$$S_{Sp(n)}(S(k\rho_n \oplus j\epsilon)) = \mathbb{Z}^{\sum_{0 \leq 2i < n} \binom{k-1}{n-2i}}.$$

The formula holds in general, with the only exception that, in case $j = 0$ and n is odd, there is one less \mathbb{Z} .

For the case $k - n$ is odd, we need to compute the top piece

$$S^{\text{alg}}(\bar{M}) = \tilde{H}_{4kn-n(2n+1)-1+j}(\bar{M}; \mathbb{L}).$$

If $j > 0$, then this is $H_{4kn-n(2n+1)-1}(\bar{N}; \mathbb{L})$, and is computed as in Proposition 6.1, plus the homology at the base point

$$H_0(\bar{N}; \pi_{4kn-n(2n+1)-1}\mathbb{L}) = L_{4kn-n(2n+1)-1}(e) = \begin{cases} \mathbb{Z}, & \text{if } n \equiv 1 \pmod{4}, \\ \mathbb{Z}_2, & \text{if } n \equiv 3 \pmod{4}, \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

If $j = 0$, then $\tilde{H}_{4kn-n(2n+1)-1+j}(\bar{M}; \mathbb{L}) = \tilde{H}_{4kn-n(2n+1)-1}(\bar{N}; \mathbb{L})$. This is the homology as computed by Proposition 6.1, where the homology at the base point is excluded. Besides the top piece, we also need to consider the case the last piece is $S(\mathbb{H}P^{k-1})$, which happens when $j = 0$ and n is even. In this case, the number of copies of \mathbb{Z} should be reduced by 1.

Theorem 6.3. *If $k \geq n$ and $k - n$ is odd, then*

$$S_{Sp(n)}(S(k\rho_n \oplus j\epsilon)) = \mathbb{Z}^{\binom{k-1}{n} + \sum_{0 \leq 2i+1 < n} \binom{k-1}{n-2i-1}},$$

with the following exceptions

1. *If $j = 0$ and n is even, then there is one less \mathbb{Z} .*
2. *If $j > 0$ and $n \equiv 1 \pmod{4}$, then there is one more \mathbb{Z} .*
3. *If $j > 0$ and $n \equiv 3 \pmod{4}$, then there is one more \mathbb{Z}_2 .*

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